INTEGRATION BY ELLIPTIC INTEGRALS.

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We will first expand the general expression

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}.$$

Let $2\cos\varphi=s+s^{-1}$. $\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=(1+e^2-es-es^{-1})^{-\frac{1}{2}(2m+1)}$

$$=(1-es)^{-\frac{1}{2}(2m+1)}(1-es^{-1})^{-\frac{1}{2}(2m+1)}=\Big(1+\frac{2m+1}{2}.es+\frac{(2m+1)(2m+3)}{2^2.2!}.e^2s^2$$

$$+\frac{(2m+1)(2m+3)(2m+5)}{2^3.3!}.e^3s^3+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4!}.e^4s^4+\ldots\ldots\Big)$$

$$\Big(1+\frac{2m+1}{2}.es^{-1}+\frac{(2m+1)(2m+3)}{2^2.2!}e^2s^{-2}+\frac{(2m+1)(2m+3)(2m+5)}{2^3.3!}e^3s^{-3}$$

$$+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4!}.e^4s^{-4}+\ldots\ldots\Big)=\Big[1+\Big(\frac{2m+1}{2}.e\Big)^2$$

$$+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2!}\Big)^2+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^3}{3!}\Big)^2+\ldots\Big]$$

$$+2\Big[\frac{2m+1}{2}.e+\Big(\frac{2m+1}{2}\Big)^2.\frac{2m+3}{2}.\frac{e^3}{2!}+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}\Big)^2.\frac{2m+5}{2}.\frac{e^5}{2!.3!}$$

$$+\ldots\Big]\Big[\frac{s+s^{-1}}{2}\Big]+2\Big[\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2!}+\Big(\frac{2m+1}{2}\Big)^2.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^4}{3!}$$

$$+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}\Big)^2.\frac{2m+5}{2}.\frac{2m+7}{2}.\frac{e^6}{2!.4!}+\ldots\Big]\Big[\frac{e^2+e^{-2}}{2}\Big]+\ldots(A).$$

$$\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=\tfrac{1}{2}P_0+P_1\cos\varphi+P_2\cos2\varphi+P_3\cos3\varphi+\ldots$$
$$+P_n\cos n\varphi.$$

When $m=0, 1, 2, 3$, etc., we get

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}A_0+A_1\cos\varphi+A_2\cos2\varphi+A_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots(B).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{3}{2}}=\tfrac{1}{2}B_0+B_1\cos\varphi+B_2\cos2\varphi+B_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots(C).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{5}{2}}=\tfrac{1}{2}C_0+C_1\cos\varphi+C_2\cos2\varphi+C_3\cos3\varphi+\ \ldots\ldots\ldots\ldots\ldots (D).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{7}{2}}=\tfrac{1}{2}D_0+D_1\cos\varphi+D_2\cos2\varphi+D_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots (E).$$

....

Let $\sin(\theta-\varphi)=e\sin\theta$, then $\tan\theta=\sin\theta/(\cos\varphi-e)$.

$$\therefore\ \cos(\theta-\varphi)\left(1-\frac{d\varphi}{d\theta}\right)=e\cos\theta.$$

$$\therefore\ \frac{d\varphi}{d\theta}=\frac{\cos(\theta-\varphi)-e\cos\theta}{\cos(\theta-\varphi)}=\frac{\sin\theta\sin\varphi+\cos\theta(\cos\varphi-e)}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\ \frac{d\varphi}{d\theta}=\frac{(\cos^2\theta\operatorname{cosec}\theta+\sin\theta)\sin\varphi}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{\sin\varphi\operatorname{cosec}\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\sqrt{\frac{(\cos\varphi-e)^2+\sin^2\varphi}{1-e^2\sin^2 6}}=\sqrt{\frac{1+e^2-2e\cos\varphi}{1-e^2\sin^2\theta}}\ \ldots\ldots\ldots (1_0).$$

Also $\sqrt{(1+e^2-2e\cos\varphi)}=\sqrt{(1-e^2\sin^2\theta)}-e\cos\theta$.

$$\therefore\ \cos\varphi=e\sin^2\theta+\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2_0).$$

$$\cos2\varphi=4e^2\sin^4\theta+4e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+1-2(1+e^2)\sin^2\theta\ .\ \ldots\ldots\ldots (3_0).$$

$$\cos3\varphi=4e^3\cos^6\theta+12e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+12e\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}-3e\sin^2\theta-3\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ldots\ldots\ldots\ldots (4_0).$$

$$\cos4\varphi=1-8e^2\sin^4\theta-16e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}-8\cos^2\theta(1-e^2\sin^2\theta)$$

$$+8e^4\sin^8\theta+32e^3\sin^6\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+48e^2\sin^4\cos^2\theta(1-e^2\sin^2\theta)$$

$$+8\cos^4\theta(1-e^2\sin^2\theta)^2+32e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}\ \ldots\ldots\ldots\ldots\ldots (5_0).$$

$$\cos5\varphi=16e^5\sin^{10}\theta+80e^4\sin^8\cos\theta\sqrt{(1-e^2\sin^2\theta)}+160e^3\sin^6\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+160e^2\sin^4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}+80e\sin^2\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$

$$+16\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}}-20e^3\sin^6\theta-60e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$-60\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)-20\cos^3\theta(1-s^2\sin^2\theta)^{\frac{3}{2}}$$

$$+5e\sin^2\theta+5\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots (6_0),$$

$$\cos 6\varphi = 32e^6\sin^{12}\theta + 192e^5\sin^{10}\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} + 480e^4\sin^8\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+640e^3\sin^6\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}} + 480e^2\sin^4\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$

$$+192e\sin^2\theta\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}} + 32\cos^6\theta(1-e^2\sin^2\theta)^3 - 48e^4\sin^8\theta$$

$$-192e^3\sin^6\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} - 288e^2\sin^4\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$-192e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}} - 48\cos^4\theta(1-e^2\sin^2\theta)^2 + 18e^2\sin^4\theta$$

$$+18\cos^2\theta(1-e^2\sin^2\theta) + 36e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)} - 1 \quad \ldots\ldots\ldots\ldots (7_0).$$

Writing (B) in the following form :

$$(1+e^2-es-es^{-1})^{-\frac{1}{2}} = \tfrac{1}{2}A_0 + \tfrac{1}{2}A_1(s+s)^{-1}) + \ldots\ldots + \tfrac{1}{2}A_n(s^n+s^{n-1}) + \ldots\ldots\ldots (8_0).$$

Differentiating (8_0) we get,

$$e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}} = A_1(1-s^{-2}) + 2A_2(s-s^{-3})$$

$$+3A_3(s^2-s^{-4}) + \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots\ldots (9_0).$$

From (8_0) and (9_0) we get,

$$\tfrac{1}{2}e(1-s^{-2})[A_0 + A_1(s+s^{-1}) + A_2(s^2+s^{-2}) + \ldots\ldots + A_n(s^n+s^{-n}) + \ldots\ldots]$$

$$= (1+e^2-es-es^{-1})[A_1(1-s^{-2}) + 2A_2(s-s^{-3}) + 3A_3(s^2-s^{-4})$$

$$+ \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots\ldots].$$

Equating coefficients of s^n we get,

$$\tfrac{1}{2}e(A_n - A_{n+2}) = (n+1)(1+e^2)A_{n+1} - e[nA_n + (n+2)A_{n+2}].$$

$$\therefore A_{n+2} = \frac{2(n+1)}{2n+3}\cdot\frac{1+e^2}{e}A_{n+1} - \frac{2n+1}{2n+3}A_n \quad \ldots\ldots\ldots\ldots\ldots (10_0).$$

$\therefore$ When A_n and A_{n+1} are known we can easily find A_{n+2}.
Multiplying (B) by $\cos\varphi$ we get,

$$\frac{\cos\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = \tfrac{1}{2}A_0\cos\varphi + \tfrac{1}{2}A_1(1+\cos 2\varphi) + \tfrac{1}{2}A_2(\cos\varphi + \cos 3\varphi) + \ldots\ldots (11_0).$$

Also $\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = F(e, \tfrac{1}{2}\pi)$ ....(1).

$$\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\, d\theta = E(e, \tfrac{1}{2}\pi) \quad\text{......................................(2).}$$

Integrating both sides of (B) and (11_0) between the limits 2π and 0, we get with the aid of (1_0), (2_0), (1), (2), the following:

$$\pi A_0 = \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = 4F(e, \tfrac{1}{2}\pi).$$

$$\therefore\ A_0 = (4/\pi)F(e, \tfrac{1}{2}\pi) \quad\text{.......................................}(12_0).$$

$$\pi A_1 = \int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4e\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{1-e^2\sin^2\theta}} + \int_0^{2\pi} \cos\theta d\theta$$

$$= \frac{4}{e}\left[\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\, d\theta\right] = \frac{4}{e}[F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)].$$

$$\therefore\ A_1 = (4/\pi e)[F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)].$$

[To be Continued.]

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION."

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[Continued from October Number.]

CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of *units*, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).